3 Sobolev Spaces

Exercise 3.1. Let $I = (0,1)$. Verify that $W^{1,\infty}(I)$ is a Banach space with respect to $\|\cdot\|_{W^{1,\infty}}$.

Exercise 3.2. Let $u \in W^{1,p}(0,1)$ with $1 < p < \infty$. Show that $\frac{u(x)}{x} \in L^1(0,1)$ if and only if $u(0) = 0$. Check that $v(x) = (1 + |\log(x)|)^{-1}$ is in $W^{1,1}(0,1)$, satisfies $v(0) = 0$ but $\frac{v(x)}{x} \notin L^1(0,1)$.

Exercise 3.3. Let $I = (0,1)$. Construct a bounded sequence in $W^{1,1}(I)$ that admits no converging subsequence in $L^\infty(I)$.

We recall the following classical theorem (see Federer's *Geometric Measure Theory*, 3.1.6).

Theorem 3.4. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitzian function. Then f is differententiable $\mathscr{L}^m$ almost everywhere, if $\mathscr{L}^m$ is the Lebesgue measure on $\mathbb{R}^m$.*

Exercise 3.5. Let $I = (0,1) \subset \mathbb{R}$. Show that the space $\mathrm{Lip}(I)$ of Lipschitzian functions $u : I \to \mathbb{R}$ coincides with the space $W^{1,\infty}(I)$.

Exercise 3.6. Let $\varphi \in \mathscr{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ be fixed different than the zero function and let $1 \leq p \leq \infty$. Set $u_n(x) = \varphi(x+n)$ and check that :

1. $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R})$,

2. there is no subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which converges strongly in $L^q(\mathbb{R})$ for any $1 \leq q \leq \infty$,

3. $u_n \underset{n\to\infty}{\longrightarrow} 0$ and $u_n' \underset{n\to\infty}{\longrightarrow} 0$ in $L^p(\mathbb{R})$ for all $1 < p < \infty$.